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Rensselaer Polytechnic Institute
Troy, New York 12181

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by

B.K. Sinha and H.F. Tiersten

Office of Naval Research
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an important influence on the surface wave velocity. In addition, perturbation calculations are performed which include the influence of the spatial variation of all flexural biasing terms and it is shown that, for substrate thickness-to-wavelength ratios well within the practical range, the spatial variation in the biasing state has an appreciable effect on the velocity of acoustic surface waves.

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ABSTRACT

A system of linear electroelastic equations for small fields superposed on a bias is applied in the determination of the velocity of acoustic surface waves in piezoelectric substrates subject to flexural biasing stresses. The influence of the biasing stresses appears in the boundary conditions as well as the differential equations. Direct calculations performed for both quartz and lithium niobate when the spatial variation of the flexural biasing state is omitted indicate that the biasing stresses in the boundary conditions have an important influence on the surface wave velocity. In addition, perturbation calculations are performed which include the influence of the spatial variation of all flexural biasing terms and it is shown that, for substrate thickness-to-wavelength ratios well within the practical range, the spatial variation in the biasing state has an appreciable effect on the velocity of acoustic surface waves.

1. Introduction

In recent years, a number of workers[1-3] have measured the change in acoustic surface wave velocity due to applied static biasing stresses, primarily for application as pressure sensors. In one[2] of the aforementioned works an analytical treatment was presented, whose validity seems questionable to us because the treatment is not related to the fundamental equations. More recently, another independent analytical treatment was presented[4], which properly considers the elastic nonlinearities in the differential equations but not in the boundary conditions.

In this paper a system of linear electroelastic equations for small fields superposed on a bias[5], which were obtained from general nonlinear rotationally invariant electroelastic equations[6], is applied in the determination of the velocity of surface waves in piezoelectric substrates subject to flexural biasing stresses. The influence of the biasing stresses appears in the boundary conditions as well as the differential equations. Although the electric, electroelastic and elastic nonlinearities are included in the basic equations, only the elastic nonlinearities are included in the calculations because the nonlinear electroelastic coefficients are not known for either lithium niobate or quartz, the two materials considered. The nonlinear electroelastic coefficients may very well be significant in lithium niobate, even though they are probably negligible in quartz. Nevertheless, numerical results are obtained for Y-Z lithium niobate and Y-cut and ST-cut quartz using the published values of the second and third-order elastic, piezoelectric and dielectric constants for these materials[7,8].

The special case of biasing stresses due to flexural loading of a simply supported plate is considered in detail and all components of the static deformation that occur in the linear equations for small fields superposed on a bias are analytically related to the applied loading. Analyses are performed for both rotated Y-cut quartz and lithium niobate subject to biasing stresses both in and normal to the propagation direction. Numerical results are presented for both Y-cut and ST-cut quartz for propagation in the digonal direction and for Y-cut lithium niobate for propagation in the Z-direction.

Finally, a perturbation equation[9] for the change in surface wave velocity under any biasing stress is employed and shown to yield results indistinguishable from those obtained from the direct calculation for the particular case of flexure considered here for wavelengths very small compared to the thickness of the plate. In fact, since the perturbation procedure can readily treat spatially varying biasing states, for which a direct calculation cannot be performed, it has a significant advantage over a direct calculation in the determination of the small changes in surface wave velocities due to biasing states. Indeed, in the case of flexural loading considered here the biasing terms actually are functions of position and this fact is ignored in the direct calculation. When spatial variation of the biasing terms is omitted in the perturbation calculation, the aforementioned agreement with the direct calculation for the thick substrate results. However, when the spatial variation in the biasing terms is included in the perturbation calculation, a readily distinguishable change from the results of the direct calculation occurs for substrate thicknesses less than about 100 wavelengths. The deviation increases significantly with decreasing thickness and becomes appreciable for thicknesses of the order of, say, 50 wavelengths, which is well within the practical range. The results of the calculation for ST-cut quartz subject to a biasing stress in the propagation direction have been compared with experiment[10] and the agreement is quite good.

2. Linear Electroelastic Equations for Small Fields Superposed on a Bias

The linear electroelastic equations for small fields superposed on a bias may be written in the form[5,9]

$$\tilde{K}_{L\gamma,L} = \rho^{\circ}\ddot{u}_{\gamma}, \quad \tilde{\mathcal{D}}_{L,L} = 0, \tag{2.1}$$

where

$$\begin{aligned} \tilde{K}_{L\gamma} &= \underset{1}{G}_{L\gamma M\nu}u_{\nu,M} + \underset{2}{G}_{ML\gamma}\tilde{\varphi}_{,M}, \\ \tilde{\mathcal{D}}_{L} &= R_{LM\alpha}u_{\alpha,M} + R_{LM}\tilde{\varphi}_{,M}, \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} &\underset{1}{G}_{L\gamma M\nu} = \underset{2}{c}_{L\gamma M\nu} + \underset{2}{\hat{c}}_{L\gamma M\nu}, \quad \underset{2}{R}_{LM} = -\epsilon_{LM} - \hat{\epsilon}_{LM}, \\ &\underset{2}{G}_{ML\gamma} = \underset{1}{R}_{ML\gamma} = e_{ML\gamma} + \hat{e}_{ML\gamma}, \\ &\underset{2}{\hat{c}}_{L\gamma M\nu} = \overset{1}{T}_{LM}\delta_{\gamma\nu} + \underset{3}{c}_{L\gamma M\nu AB}\overset{1}{E}_{AB} + \underset{2}{c}_{L\gamma KM}w_{\nu,K} \\ &\qquad + \underset{2}{c}_{LKM\nu}w_{\gamma,K} - \underset{1}{k}_{AL\gamma M\nu}\overset{1}{\hat{\varphi}}_{,A} + \overset{1}{g}_{L\gamma M\nu}, \\ &\hat{e}_{ML\gamma} = -\underset{1}{k}_{ML\gamma BC}\overset{1}{E}_{BC} - e_{MLK}w_{\gamma,K} - b_{AML\gamma}\overset{1}{\hat{\varphi}}_{,A} + \overset{2}{g}_{ML\gamma}, \\ &\hat{\epsilon}_{LM} = b_{LMCD}\overset{1}{E}_{CD} - \underset{2}{\chi}_{LMC}\overset{1}{\hat{\varphi}}_{,C} - 2\epsilon_{o}\overset{1}{J}\overset{1}{E}_{ML}, \end{aligned} \tag{2.3}$$

and $\overset{1}{g}_{L\gamma M\nu}$ and $\overset{2}{g}_{ML\gamma}$ are lengthy expressions[5,9] involving the biasing electric field and static deformation, which vanish when the biasing electric field vanishes and are not of interest here. Consequently, we do not present them here. Equations (2.1) constitute the stress equations of motion and charge equation of electrostatics referred to the position coordinates of material points before the static deformation is applied, which are called reference coordinates and are denoted X_M. Equations (2.2) are the linear electroelastic constitutive equations and Eqs. (2.3) contain the definitions of the effective coefficients defined therein. In (2.1) - (2.3) $\tilde{K}_{L\gamma}$, u_{γ} and $\tilde{\mathcal{D}}_{L}$ denote the components of the small field Piola-Kirchhoff stress tensor which is asymmetric, mechanical displacement vector, and reference electric displacement vector[5], respectively; ρ° and $\tilde{\varphi}$ denote the reference mass density and small field electric potential, respectively; $c_{2L\gamma M\nu}$, $e_{ML\gamma}$ and ε_{LM} denote the second order elastic, piezoelectric and dielectric constants, respectively, which are the constants that occur in the ordinary linear theory of piezoelectricity.

Before discussing the remaining quantities in (2.3) we note that in a motion the present position of material points $\tilde{y}$ may be written

$$\tilde{y}(X_L, t) = \tilde{X} + \tilde{w}(X_L) + \tilde{u}(X_L, t)\,, \tag{2.4}$$

where $\tilde{w}$ denotes the displacement due to the applied static loading. In addition, we note that the electric potentials $\hat{\varphi}$ in the body and $\hat{\psi}$ in free space may be written in the respective forms

$$\hat{\varphi}(X_L, t) = \hat{\varphi}^1(X_L) + \tilde{\hat{\varphi}}(X_L, t)\,, \quad \hat{\psi}(X_L, t) = \hat{\psi}^1(X_L) + \tilde{\hat{\psi}}(X_L, t)\,, \tag{2.5}$$

where $\hat{\varphi}^1$ and $\hat{\psi}^1$ denote the static biasing electric potentials in the body and in free space. In (2.3) T^1_{LM} and E^1_{AB} denote the components of the static biasing stress and strain, respectively, and J^1 is the Jacobian of the static deformation. The biasing variables satisfy the appropriate static equations given in (66) - (72) of Ref.5, or the equivalent equations using reference coordinates as independent variables. Since we are interested in small biasing strains only, we have

$$E^1_{AB} = \frac{1}{2}\,(w_{A,B} + w_{B,A})\,, \quad J^1 \approx 1\,, \tag{2.6}$$

$$T^1_{LM} = c_{2LMRS}\, w_{R,S} + e_{RLM}\,\hat{\varphi}^1_{,M}\,. \tag{2.7}$$

The quantities $c_{3L\gamma M\alpha AB}$, $b_{AML\gamma}$, χ_{2LMC} and $k_{1ML\gamma BC}$ denote the third-order elastic, electrostrictive, third-order electric permeability and first-order electroelastic constants, respectively, and ε_o denotes the electric permittivity of free space. For obvious reasons the notation employed here is designed to be consistent with the notation of Refs.5 and 9. The carets over many variables have been employed here because we consistently use the reference coordinates as independent variables. In particular the Greek subscripts are employed in order to be consistent with the notation of Refs. 5 and 9, in which Greek subscripts were consistently employed to denote the intermediate coordinates ξ_α. However, the Greek subscripts are not really important for any of the operations performed here and may be used interchangeably with the capital Latin subscripts, which denote reference coordinates X_L. The cycles above variables also have been introduced for consistency with Refs. 5 and 9. We have employed Cartesian tensor notation and introduced vector notation in Eq. (2.4) only, and the convention that a comma followed by an index denotes partial differentiation with respect to a reference coordinate, the dot notation for partial differentiation with respect to time and the summation convention for repeated tensor indices.

To these equations we must adjoin the boundary conditions[5,9]

$$N_L(\tilde{K}_{L\gamma} - \tilde{K}^f_{L\gamma}) = 0, \quad N_L(\tilde{\mathcal{D}}_L - \tilde{\mathcal{D}}^f_L) = 0,$$
$$\tilde{\hat{\varphi}} = \hat{\psi}^1_{,L} u_L + \tilde{\hat{\psi}}_{,L} w_L + \tilde{\hat{\psi}} \tag{2.8}$$

where w_L and u_L in $(2.8)_3$, respectively, denote the static biasing and small field dynamic components of the mechanical displacement at the surface of the solid, N_L denotes the unit normal to the reference position of the surface and

$$\tilde{K}^f_{L\gamma} = {}_1G^f_{L\gamma\beta} u_\beta + {}_2G^f_{L\gamma M\alpha} u_{\alpha,M} + {}_3G^f_{L\gamma M} \tilde{\hat{\psi}}_{,M},$$
$$\tilde{\mathcal{D}}^f_L = {}_1R^f_{L\beta} u_\beta + {}_2R^f_{LM\gamma} u_{\gamma,M} - \epsilon_o \tilde{\hat{\psi}}_{,L} \tag{2.9}$$

The coefficients ${}_1G^f_{L\gamma\beta}$, ${}_2G^f_{L\gamma M\alpha}$, ${}_3G^f_{LM\gamma}$, ${}_1R^f_{L\beta}$ and ${}_2R^f_{LM\gamma}$ are lengthy expressions[5,9] involving the biasing electric field and static deformation, which vanish when the biasing electric field vanishes and are not of interest here. Consequently, we do not present them here. In free space $\tilde{\hat{\psi}}$ satisfies Laplaces equation[5,9], i.e.,

$$\tilde{\hat{\psi}}_{,KK} = 0 . \tag{2.10}$$

Since the nonlinear electroelastic coefficients are not known for either lithium niobate or quartz, or to our knowledge for any material, only the elastic nonlinearities will be included in the remainder of the analysis and in the calculations. However, we wish to emphasize the fact that we could equally readily have included the electroelastic nonlinearities if the coefficients were known. In the case of a purely elastic bias Eqs. (2.1) and (2.2) remain unchanged, Eq. $(2.3)_4$ reduces to the form

$$\hat{c}_{2L\gamma M\nu} = T^{1}_{LM}\delta_{\gamma\nu} + c_{3L\gamma M\nu AB}E^{1}_{AB} + c_{2L\gamma KM}w_{\nu,K} + c_{2LKM\nu}w_{\gamma,K} \tag{2.11}$$

and Eqs. $(2.3)_5$ and $(2.3)_6$ are replaced by

$$\hat{e}_{ML\gamma} = 0 , \quad \hat{\epsilon}_{LM} = 0 . \tag{2.12}$$

In the case of the purely elastic bias the boundary conditions in (2.8) reduce to

$$N_L\tilde{K}_{L\gamma} = 0 , \quad N_L\tilde{\mathcal{D}}_L = - N_L\epsilon_o\tilde{\hat{\psi}}_{,L} , \quad \tilde{\hat{\varphi}} = \tilde{\hat{\psi}} , \tag{2.13}$$

and the free space equation (2.10) remains unchanged.

3. Static Flexural Loading

Since an acoustic surface wave is sharply confined to the vicinity of the surface of the substrate in which it propagates and in static flexure of a thin plate the magnitude of the stress is maximum at the outer fibers, it is advantageous to employ flexural loading structures in order to maximize the influence on the surface wave velocity for a given loading. Consequently, in this section we briefly discuss cylindrical flexure of thin plates for the type of loading shown in Fig.1. The elementary theory of cylindrical flexure of thin plates, which is known to be extremely accurate when applicable, rests on a few critical simplifying assumptions. In most treatments of elementary flexure, the theory is not presented in a way that readily indicates the relation between the flexural variables employed and the mechanical displacement vector of the three-dimensional theory, which is needed here for the static biasing terms in Eq. (2.11), nor is the theory discussed in the case of anisotropy. However, Mindlin[11] has presented a derivation of the equations of the theory of flexure of thin anisotropic plates in a manner that clearly indicates the relation between the plate variables and the mechanical displacement vector of the three-dimensional equations. The brief discussion presented for the special case of interest here is based on material in Ref. 11.

In the case of cylindrical flexure in the x_1-direction with restrained motion in the x_3-direction the aforementioned critical assumptions are that the static displacement field w_γ may be written in the form

$$w_1 = z_2 w_1^{(1)}(z_1), \quad w_2 = w_2^{(o)}(z_1) + z_2^2 w_2^{(2)}(z_1), \quad w_3 = 0, \tag{3.1}$$

the zero order[11] shearing strain $S_{12}^{(o)}$ vanishes while the zero order shearing stress $T_{12}^{(o)}$ is related to the bending moment $T_{11}^{(1)} \equiv M$ by the usual static equation

$$T_{12}^{(o)} = M_{,1}, \tag{3.2}$$

and the first order thickness stress $T_{22}^{(1)}$ vanishes. The conditions $S_{12}^{(o)} = 0$ and $T_{22}^{(1)} = 0$, respectively, yield the important relations

$$w_1^{(1)} = -\, w_{2,1}^{(o)}, \quad w_2^{(2)} = -\, (c_{12}/2c_{22}) w_{1,1}^{(1)} . \tag{3.3}$$

Substituting from (3.3) into the constitutive relation for M, we obtain

$$M = -\, (2h^3/3) c_{11}^{*} w_{2,11}^{(o)}, \tag{3.4}$$

where

$$c^*_{11} = c_{11} - c^2_{12}/c_{22}, \tag{3.5}$$

and it is to be emphasized that these equations hold for arbitrary orientations of x_1 in the anisotropic case. From the loading shown in Fig.1, in the region in which the surface wave velocity is measured, i.e., $|z_2| < b$, we have

$$M = - Pa, \tag{3.6}$$

where P is the load per linear dimension, which with (3.2) indicates that $T^{(o)}_{12} = 0$. Since the slope of the deflection curve vanishes at $z_1 = 0$, from (3.4) we obtain

$$w^{(o)}_{2,1} = - z_1 (3/2h^3) M/c^*_{11}, \tag{3.7}$$

which relates the loading to the slope of the deflection curve. From (3.1) and (3.3) we obtain the important relation

$$w_{2,2} = - (c_{12}/c_{22}) w_{1,1}. \tag{3.8}$$

Now, from (3.1), (3.3) and (3.7), we obtain

$$w_{1,1} = z_2 (3/2h^3) M/c^*_{11}. \tag{3.9}$$

Moreover, from (3.1), (3.3) and (3.7), we find

$$w_{1,2} = z_1 \frac{3M}{2h^3 c^*_{11}}, \quad w_{2,1} = - z_1 \frac{3M}{2h^3 c^*_{11}}, \tag{3.10}$$

and all other three-dimensional displacement gradients vanish. From (2.6), $(3.1)_3$ and (3.7) - (3.10), we obtain

$$E^1_{11} = z_2 \frac{3M}{2h^3 c^*_{11}}, \quad E^1_{22} = - z_2 \frac{3M}{2h^3 c^*_{11}} \frac{c_{12}}{c_{22}}, \quad E^1_{12} = 0, \quad E^1_{3K} = 0, \tag{3.11}$$

for the three-dimensional strain field. In addition, from the usual three-dimensional linear anisotropic stress-strain relations and (3.11), we have

$$T^1_{11} = z_2 \frac{3M}{2h^3}\,,\quad T^1_{33} = \frac{T^1_{11}}{c^*_{11}}\left(c_{13} - \frac{c_{23}c_{12}}{c_{22}}\right),\quad T^1_{31} = \frac{T^1_{11}}{c^*_{11}}\left(c_{15} - \frac{c_{25}c_{12}}{c_{22}}\right) \tag{3.12}$$

which are the nonvanishing components of stress resulting from (3.11). Note that at the top fiber $z_2 = -h$, T^1_{11} is a maximum and has the value

$$T^{1m}_{11} = 3M/2h^2\,. \tag{3.13}$$

In the case of cylindrical flexure in the x_3-direction, the analytical treatment is the same except for the fact that the index 3 replaces 1 throughout. Under these circumstances the pertinent displacement gradients and strain components take the respective forms

$$w_{3,2} = z_3 \frac{3M}{2h^3 c^*_{33}}\,,\quad w_{2,3} = -\,z_3 \frac{3M}{2h^3 c^*_{33}}\,, \tag{3.14}$$

$$E^1_{33} = z_2 \frac{3M}{2h^3 c^*_{33}}\,,\quad E^1_{22} = -\,z_2 \frac{3M}{2h^3 c^*_{33}}\frac{c_{32}}{c_{22}}\,,\quad E^1_{32} = 0\,,\quad E^1_{1K} = 0\,, \tag{3.15}$$

in place of the forms in (3.10) and (3.11) and where

$$c^*_{33} = c_{33} - c^2_{32}/c_{22}\,. \tag{3.16}$$

In addition, in place of (3.12) and (3.13) we have, respectively,

$$T^1_{33} = z_2 \frac{3M}{2h^3}\,,\quad T^1_{11} = \frac{T^1_{33}}{c^*_{33}}\left(c_{13} - \frac{c_{12}c_{32}}{c_{22}}\right),\quad T^1_{13} = \frac{T^1_{33}}{c^*_{33}}\left(c_{35} - \frac{c_{25}c_{32}}{c_{22}}\right), \tag{3.17}$$

$$T^{1m}_{33} = 3M/2h^2\,. \tag{3.18}$$

Thus, depending on the loading situation, we use either Eqs. (3.10) - (3.12) or Eqs. (3.14), (3.15) and (3.17) in the static bias equation (2.11).

4. Application to Rotated Y-Cut Quartz and Y-Cut Lithium Niobate

For the case of static flexural biasing stresses both in and normal to the propagation (X_1)-direction, for rotated Y-cut quartz, Eqs. (2.1) and (2.2), with

(2.11), the appropriate relations in Sec. 3 and

$$w_{2,2} = -(c_{32}/c_{22})w_{3,3}, \tag{4.1}$$

take the form[12,13]

$$\begin{aligned}
&c_{11}u_{1,11} + (c_{12}+c_{66})u_{2,12} + (c_{14}+c_{56})u_{3,12} + c_{66}u_{1,22} + e_{11}\tilde{\hat{\varphi}}_{,11} + e_{26}\tilde{\hat{\varphi}}_{,22} \\
&\quad + E^1_{11}[(c^*_{11} + 2c_{11} + c_{111} + \alpha c_{112})u_{1,11} + (2c_{66} + c_{166} + \alpha c_{266})u_{1,22} \\
&\quad + (c_{12} + c_{66} + c_{112} + c_{166} + \alpha(c_{12} + c_{66} + c_{122} + c_{266}))u_{2,12} + (c_{14} + c_{56} \\
&\quad + c_{114} + c_{156} + \alpha(c_{124} + c_{256}))u_{3,12}] + w_{1,2}[c_{66}u_{2,11} + c_{22}u_{2,22} + 2(c_{12} \\
&\quad + c_{66})u_{1,12} + c_{56}u_{3,11} + c_{24}u_{3,22}] + w_{2,1}[c_{11}u_{2,11} + c_{66}u_{2,22}] \\
&\quad + E^1_{33}[(c_{13} + c_{113} + \beta(c_{12} + c_{112}))u_{1,11} + (c_{123} + c_{366} + \beta(c_{12} + c_{122} + c_{66} \\
&\quad + c_{266}))u_{2,12} + (c_{14} + c_{56} + c_{134} + c_{356} + \beta(c_{124} + c_{256}))u_{3,12} + (c_{366} + \\
&\quad + \beta c_{266})u_{1,22}] + w_{3,2}(c_{12} + c_{66})u_{3,12} + w_{2,3}(c_{14} + c_{56})u_{2,12} = \rho\ddot{u}_1,
\end{aligned}$$

$$\begin{aligned}
&c_{56}u_{3,11} + (c_{12}+c_{66})u_{1,12} + c_{66}u_{2,11} + c_{22}u_{2,22} + c_{24}u_{3,22} + (e_{26} + e_{12})\tilde{\hat{\varphi}}_{,12} \\
&\quad + E^1_{11}[(c_{12} + c_{66} + c_{112} + c_{166} + \alpha(c_{12} + c_{66} + c_{122} + c_{266}))u_{1,12} + (c^*_{11} + c_{166} \\
&\quad + \alpha(2c_{66} + c_{266}))u_{2,11} + (c_{122} + \alpha(2c_{22} + c_{222}))u_{2,22} + (c_{156} + \alpha(c_{56} \\
&\quad + c_{256}))u_{3,11} + (c_{124} + \alpha(c_{24} + c_{224}))u_{3,22}] + w_{1,2}[c_{66}u_{1,11} + c_{22}u_{1,22}] \\
&\quad + w_{2,1}[c_{11}u_{1,11} + c_{66}u_{1,22} + 2(c_{12} + c_{66})u_{2,12} + (c_{14} + c_{56})u_{3,12}] \\
&\quad + E^1_{33}[(c_{13} + c_{366} + \beta(c_{12} + 2c_{66} + c_{266}))u_{2,11} + (c_{123} + c_{366} + \beta(c_{12} + c_{66} + c_{122} \\
&\quad + c_{266}))u_{1,12} + (c_{56} + c_{356} + \beta(c_{56} + c_{256}))u_{3,11} + (c_{223} + \beta(2c_{22} + c_{222}))u_{2,22} \\
&\quad + (c_{24} + c_{234} + \beta(c_{24} + c_{224}))u_{3,22}] + w_{3,2}[c_{22}u_{3,22} + c_{66}u_{3,11}] + w_{2,3}[(c_{14} \\
&\quad + c_{56})u_{1,12} + 2c_{56}u_{2,11} + 2c_{24}u_{2,22} + c_{55}u_{3,11} + c_{44}u_{3,22}] = \rho\ddot{u}_2,
\end{aligned}$$

$$
\begin{aligned}
& c_{55}u_{3,11} + (c_{56} + c_{14})u_{1,12} + c_{56}u_{2,11} + c_{24}u_{2,22} + c_{44}u_{3,22} + (e_{25} + e_{14})\tilde{\hat{\varphi}}_{,12} \\
& + E^1_{11}[(c_{14} + c_{56} + c_{114} + c_{156} + \alpha(c_{124} + c_{256}))u_{1,12} + (c_{156} + \alpha(c_{56} + c_{256}))u_{2,11} \\
& + (c_{124} + \alpha(c_{24} + c_{224}))u_{2,22} + (c^*_{11} + c_{155} + \alpha c_{255})u_{3,11} + (c_{144} + \alpha c_{244})u_{3,22}] \\
& + w_{1,2}[c_{56}u_{1,11} + c_{24}u_{1,22}] + w_{2,1}(c_{14} + c_{56})u_{2,12} + E^1_{33}[(c_{14} + c_{56} + c_{134} \\
& + c_{356} + \beta(c_{124} + c_{256}))u_{1,12} + (c_{56} + c_{356} + \beta(c_{56} + c_{256}))u_{2,11} + (c_{24} + c_{234} \\
& + \beta(c_{24} + c_{224}))u_{2,22} + (c_{13} + \beta c_{12} + 2c_{55} + c_{355} + \beta c_{255})u_{3,11} + (2c_{44} + c_{344} \\
& + \beta c_{244})u_{3,22}] + w_{3,2}[(c_{12} + c_{66})u_{1,12} + c_{66}u_{2,11} + c_{22}u_{2,22} + 2c_{56}u_{3,11} \\
& \qquad + 2c_{24}u_{3,22}] + w_{2,3}[c_{44}u_{2,22} + c_{55}u_{2,11}] = \rho\ddot{u}_3 \,, \\
& e_{11}u_{1,11} + (e_{12} + e_{26})u_{2,12} + (e_{14} + e_{25})u_{3,12} + e_{26}u_{1,22} \\
& \qquad - \epsilon_{11}\tilde{\hat{\varphi}}_{,11} - \epsilon_{22}\tilde{\hat{\varphi}}_{,22} = 0 \,, \qquad (4.2)
\end{aligned}
$$

where

$$\alpha = - c_{12}/c_{22} \,, \quad \beta = - c_{32}/c_{22} \,. \qquad (4.3)$$

From (2.13), with (2.10), for straight-crested waves with propagation wave-number ξ, we have the boundary conditions

$$\tilde{K}_{2\gamma} = 0 \,, \quad \tilde{\mathcal{D}}_2 = - \epsilon_o \xi \tilde{\hat{\varphi}} \,, \quad \text{at } X_2 = 0 \,, \qquad (4.4)$$

and from (2.2) and (2.11) we obtain[12,13] the pertinent constitutive relations

$$
\begin{aligned}
\tilde{K}_{21} = {} & c_{56}u_{3,1} + c_{66}(u_{1,2} + u_{2,1}) + e_{26}\tilde{\hat{\varphi}}_{,2} + E^1_{11}[(2c_{66} + c_{166} + \alpha c_{266})u_{1,2} \\
& + (c_{66} + c_{166} + \alpha(c_{66} + c_{266}))u_{2,1} + (c_{56} + c_{156} + \alpha c_{256})u_{3,1}] + w_{1,2}[(c_{12} \\
& + c_{66})u_{1,1} + c_{22}u_{2,2} + c_{24}u_{3,2}] + w_{2,1}c_{66}u_{2,2} + E^1_{33}[(c_{366} + \beta c_{266})u_{1,2} \\
& + (c_{366} + \beta(c_{66} + c_{266}))u_{2,1} + (c_{56} + c_{356} + \beta c_{256})u_{3,1}] + w_{3,2}c_{66}u_{3,1} + w_{2,3}c_{56}u_{2,1} \,,
\end{aligned}
$$

$$c_{55}u_{3,11} + (c_{56}+c_{14})u_{1,12} + c_{56}u_{2,11} + c_{24}u_{2,22} + c_{44}u_{3,22} + (e_{25}+e_{14})\tilde{\hat{\varphi}}_{,12}$$
$$+ E^1_{11}[(c_{14}+c_{56}+c_{114}+c_{156}+\alpha(c_{124}+c_{256}))u_{1,12} + (c_{156}+\alpha(c_{56}+c_{256}))u_{2,11}$$
$$+(c_{124}+\alpha(c_{24}+c_{224}))u_{2,22} + (c^*_{11}+c_{155}+\alpha c_{255})u_{3,11} + (c_{144}+\alpha c_{244})u_{3,22}]$$
$$+ w_{1,2}[c_{56}u_{1,11}+c_{24}u_{1,22}] + w_{2,1}(c_{14}+c_{56})u_{2,12} + E^1_{33}[(c_{14}+c_{56}+c_{134}$$
$$+ c_{356}+\beta(c_{124}+c_{256}))u_{1,12} + (c_{56}+c_{356}+\beta(c_{56}+c_{256}))u_{2,11} + (c_{24}+c_{234}$$
$$+ \beta(c_{24}+c_{224}))u_{2,22} + (c_{13}+\beta c_{12}+2c_{55}+c_{355}+\beta c_{255})u_{3,11} + (2c_{44}+c_{344}$$
$$+ \beta c_{244})u_{3,22}] + w_{3,2}[(c_{12}+c_{66})u_{1,12} + c_{66}u_{2,11} + c_{22}u_{2,22} + 2c_{56}u_{3,11}$$
$$+ 2c_{24}u_{3,22}] + w_{2,3}[c_{44}u_{2,22} + c_{55}u_{2,11}] = \rho\ddot{u}_3 ,$$

$$e_{11}u_{1,11} + (e_{12}+e_{26})u_{2,12} + (e_{14}+e_{25})u_{3,12} + e_{26}u_{1,22}$$
$$- \varepsilon_{11}\tilde{\hat{\varphi}}_{,11} - \varepsilon_{22}\tilde{\hat{\varphi}}_{,22} = 0 , \qquad (4.2)$$

where

$$\alpha = - c_{12}/c_{22}, \quad \beta = - c_{32}/c_{22} . \qquad (4.3)$$

From (2.13), with (2.10), for straight-crested waves with propagation wave-number ξ, we have the boundary conditions

$$\tilde{K}_{2\gamma} = 0, \quad \tilde{\mathcal{D}}_2 = - \varepsilon_o\xi\tilde{\hat{\varphi}}, \quad \text{at } X_2 = 0, \qquad (4.4)$$

and from (2.2) and (2.11) we obtain[12,13] the pertinent constitutive relations

$$\tilde{K}_{21} = c_{56}u_{3,1} + c_{66}(u_{1,2}+u_{2,1}) + e_{26}\tilde{\hat{\varphi}}_{,2} + E^1_{11}[(2c_{66}+c_{166}+\alpha c_{266})u_{1,2}$$
$$+ (c_{66}+c_{166}+\alpha(c_{66}+c_{266}))u_{2,1} + (c_{56}+c_{156}+\alpha c_{256})u_{3,1}] + w_{1,2}[(c_{12}$$
$$+ c_{66})u_{1,1} + c_{22}u_{2,2} + c_{24}u_{3,2}] + w_{2,1}c_{66}u_{2,2} + E^1_{33}[(c_{366}+\beta c_{266})u_{1,2}$$
$$+ (c_{366}+\beta(c_{66}+c_{266}))u_{2,1} + (c_{56}+c_{356}+\beta c_{256})u_{3,1}] + w_{3,2}c_{66}u_{3,1} + w_{2,3}c_{56}u_{2,1} ,$$

$$\tilde{K}_{22} = c_{12}u_{1,1} + c_{22}u_{2,2} + c_{24}u_{3,2} + e_{12}\tilde{\hat{\varphi}}_{,1} + E^1_{11}[(c_{12} + c_{112} + \alpha(c_{12} + c_{122}))u_{1,1}$$

$$+ (c_{122} + \alpha(2c_{22} + c_{222}))u_{2,2} + (c_{124} + \alpha(c_{24} + c_{224}))u_{3,2}] + w_{1,2}c_{22}u_{1,2}$$

$$+ w_{2,1}[(c_{12} + c_{66})u_{2,1} + c_{66}u_{1,2} + c_{56}u_{3,1}] + E^1_{33}[(c_{123} + \beta(c_{12} + c_{122}))u_{1,1}$$

$$+ (c_{223} + \beta(2c_{22} + c_{222}))u_{2,2} + (c_{24} + c_{234} + \beta(c_{24} + c_{224}))u_{3,2}] + w_{3,2}c_{22}u_{3,2}$$

$$+ w_{2,3}[c_{14}u_{1,1} + 2c_{24}u_{2,2} + c_{44}u_{3,2}] ,$$

$$\tilde{K}_{23} = c_{14}u_{1,1} + c_{24}u_{2,2} + c_{44}u_{3,2} + e_{14}\tilde{\hat{\varphi}}_{,1} + E^1_{11}[(c_{14} + c_{114} + \alpha c_{124})u_{1,1} + (c_{124}$$

$$+ \alpha(c_{24} + c_{224}))u_{2,2} + (c_{144} + \alpha c_{244})u_{3,2}] + w_{1,2}c_{24}u_{1,2} + w_{2,1}c_{14}u_{2,1}$$

$$+ E^1_{33}[(c_{14} + c_{134} + \beta c_{124})u_{1,1} + (c_{24} + c_{234} + \beta(c_{24} + c_{224}))u_{2,2} + (2c_{44}$$

$$+ c_{344} + \beta c_{244})u_{3,2}] + w_{3,2}[c_{22}u_{2,2} + c_{12}u_{1,1} + 2c_{24}u_{3,2}] + w_{2,3}c_{44}u_{2,2} ,$$

$$\tilde{\mathcal{D}}_2 = e_{25}u_{3,1} + e_{26}(u_{1,2} + u_{2,1}) - \epsilon_{22}\tilde{\hat{\varphi}}_{,2} . \tag{4.5}$$

For the direct calculations performed in this section we take E^1_{11} and E^1_{33} to be constant with respect to the decay length of the surface wave. This is a reasonably accurate assumption for thick substrates for which the surface wave decays rapidly compared to the variation of the flexural stress, but not for relatively thin substrates. In addition, in the direct calculation performed in this section we ignore the variable displacement gradient terms $w_{1,2}$, $w_{2,1}$ and $w_{3,2}$, $w_{2,3}$ in (3.10) and (3.14), which do not give rise to strains, because of their dependence on the position coordinate X_1. The influence of this latter assumption is not too great because the neglected terms do not contain the third order elastic constants, which are considerably larger than the second order elastic constants. Calculations are performed for cylindrical flexure either in the X_1-direction or in the X_3-direction only, not in both directions simultaneously. Consequently, in the calculations performed the biasing terms containing either the index 3 or 1 in (4.2) and (4.5) vanish.

A solution satisfying (4.2) and (4.4), with (4.5), may be taken in the usual form

$$u_\gamma = \sum_{m=1}^{4} C^{(m)} A_\gamma^{(m)} e^{i\beta_m \xi X_2} e^{i\xi(X_1 - Vt)} ,$$

$$\hat{\tilde{\varphi}} = \sum_{m=1}^{4} C^{(m)} B^{(m)} e^{i\beta_m \xi X_2} e^{i\xi(X_1 - Vt)} , \tag{4.6}$$

where V, β_m, $A_\gamma^{(m)}$, $B^{(m)}$ and $C^{(m)}$ are determined numerically so that (4.2) and (4.4), with (4.5), are satisfied in the usual manner[14-16]. Calculations have been performed for ST-cut and Y-cut quartz[17,18] in X_1-flexure and the results are plotted as the line labeled a in Figs.2 and 3, respectively. In addition, we have performed calculations for ST-cut quartz in X_1-flexure when the biasing terms, i.e., those proportional to E^1_{11}, have been ignored in the boundary conditions (4.4), with (4.5), and the resulting line is labeled a^* in Fig.2. It is clear from the figure that the difference is quite great and that, consequently, in general the nonlinearities in the boundary conditions are quite important. Calculations have been performed for X_3-flexure also, and the results appear as the line labeled b in Figs.2 and 3 for ST-cut and Y-cut quartz, respectively. No line labeled a^* appears in Fig.3.

For the case of static flexural biasing stresses both in and normal to the propagation (X_3)-direction for Y-cut lithium niobate, Eqs.(2.1) and (2.2), with (2.11), the appropriate relations in Sec.3 and (4.1) take the form[12,13]

$$
\begin{aligned}
&c_{22}u_{2,22} + (c_{23}+c_{44})u_{3,23} + 2c_{24}u_{2,23} + c_{24}u_{3,22} + c_{44}u_{2,33} + e_{22}\tilde{\hat{\varphi}}_{,22} + (e_{32}+e_{24})\tilde{\hat{\varphi}}_{,23} \\
&+ E^{1}_{33}[(c_{223} + \beta(2c_{22}+c_{222}))u_{2,22} + (2c_{234} + \beta(4c_{24}+2c_{224}))u_{2,23} + (c^{*}_{33} + c_{344} \\
&+ \beta(2c_{44}+c_{244}))u_{2,33} + (c_{24}+c_{234}+\beta(c_{24}+c_{224}))u_{3,22} + (c_{23}+c_{44}+c_{233}+c_{344} \\
&+ \beta(c_{23}+c_{44}+c_{223}+c_{244}))u_{3,23} + (c_{34}+c_{334}+\beta(c_{34}+c_{234}))u_{3,33}] \\
&+ w_{3,2}[c_{22}u_{3,22} + 2c_{24}u_{3,23} + c_{44}u_{3,33}] + w_{2,3}[2c_{24}u_{2,22} + 2(c_{23}+c_{44})u_{2,23} \\
&+ 2c_{34}u_{2,33} + c_{44}u_{3,22} + 2c_{34}u_{3,23} + c_{33}u_{3,33}] + E^{1}_{11}[(c_{122}+\alpha(2c_{22}+c_{222}))u_{2,22} \\
&+ (2c_{124}+\alpha(4c_{24}+2c_{224}))u_{2,23} + (c_{13}+c_{144}+\alpha(c_{23}+2c_{44}+c_{244}))u_{2,33} + (c_{124} \\
&+ \alpha(c_{24}+c_{224}))u_{3,22} + (c_{123}+c_{144}+\alpha(c_{23}+c_{44}+c_{223}+c_{244}))u_{3,23} + (c_{134} \\
&\qquad\qquad + \alpha(c_{34}+c_{234}))u_{3,33}] = \rho\ddot{u}_2 ,
\end{aligned}
$$

$$
\begin{aligned}
&c_{24}u_{2,22} + (c_{23}+c_{44})u_{2,23} + c_{44}u_{3,22} + c_{33}u_{3,33} + e_{24}\tilde{\hat{\varphi}}_{,22} + e_{33}\tilde{\hat{\varphi}}_{,33} + E^{1}_{33}[(c_{24} \\
&+ c_{234} + \beta(c_{24}+c_{224}))u_{2,22} + (c_{23}+c_{44}+c_{233}+c_{344}+\beta(c_{23}+c_{44}+c_{223} \\
&+ c_{244}))u_{2,23} + (c_{34}+c_{334}+\beta(c_{34}+c_{234}))u_{2,33} + (2c_{44}+c_{344}+\beta c_{244})u_{3,22} \\
&+ (4c_{34}+2c_{334}+2\beta c_{234})u_{3,23} + (2c_{33}+c_{333}+c^{*}_{33}+\beta c_{233})u_{3,33} + w_{3,2}[c_{22}u_{2,22} \\
&+ 2c_{24}u_{2,23} + c_{44}u_{2,33} + 2c_{24}u_{3,22} + 2(c_{23}+c_{44})u_{3,23} + 2c_{34}u_{3,33}] + w_{2,3}[c_{44}u_{2,22} \\
&+ 2c_{34}u_{2,23} + c_{33}u_{2,33}] + E^{1}_{11}[(c_{124}+\alpha(c_{24}+c_{224}))u_{2,22} + (c_{123}+c_{144}+\alpha(c_{23} \\
&+ c_{44}+c_{223}+c_{244}))u_{2,23} + (c_{134}+\alpha(c_{34}+c_{234}))u_{2,23} + (c_{144}+\alpha c_{244})u_{3,22} \\
&+ 2(c_{134}+\alpha c_{234})u_{3,23} + (c_{13}+\alpha c_{23}+c_{133}+\alpha c_{233})u_{3,33}] = \rho\ddot{u}_3 ,
\end{aligned}
$$

$$
e_{22}u_{2,22} + (e_{24}+e_{32})u_{2,23} + e_{24}u_{3,22} + e_{33}u_{3,33} - \epsilon_{22}\tilde{\hat{\varphi}}_{,22} - \epsilon_{33}\tilde{\hat{\varphi}}_{,33} = 0 , \qquad (4.7)
$$

where α and β are given in (4.3). Again the boundary conditions are given by (4.4) and from (2.2) and (2.11) we obtain[12,13] the pertinent constitutive equations

$$\tilde{K}_{22} = c_{22}u_{2,2} + c_{23}u_{3,3} + c_{24}(u_{2,3} + u_{3,2}) + e_{22}\tilde{\hat{\varphi}}_{,2} + e_{32}\tilde{\hat{\varphi}}_{,3} + E^1_{33}[(c_{223} + \beta(2c_{22} + c_{222}))u_{2,2} + (c_{234} + \beta(2c_{24} + c_{224}))u_{2,3} + (c_{24} + c_{234} + \beta(c_{24} + c_{224}))u_{3,2} + (c_{23} + c_{233} + \beta(c_{23} + c_{223}))u_{3,3}] + w_{3,2}[c_{22}u_{3,2} + c_{24}u_{3,3}] + w_{2,3}[2c_{24}u_{2,2} + (c_{23} + c_{44})u_{2,3} + c_{44}u_{3,2} + c_{34}u_{3,3}] + E^1_{11}[(c_{122} + \alpha(2c_{22} + c_{222}))u_{2,2} + (c_{124} + \alpha(2c_{24} + c_{224}))u_{2,3} + (c_{124} + \alpha(c_{24} + c_{224}))u_{3,2} + (c_{123} + \alpha(c_{23} + c_{223}))u_{3,3}] ,$$

$$\tilde{K}_{23} = c_{24}u_{2,2} + c_{34}u_{3,3} + c_{44}(u_{2,3} + u_{3,2}) + e_{24}\tilde{\hat{\varphi}}_{,2} + E^1_{33}[(c_{24} + c_{234} + \beta(c_{24} + c_{224}))u_{2,2} + (c_{44} + c_{344} + \beta(c_{44} + c_{244}))u_{2,3} + (2c_{44} + c_{344} + \beta c_{244})u_{3,2} + (2c_{34} + c_{334} + \beta c_{234})u_{3,3}] + w_{3,2}[c_{22}u_{2,2} + c_{24}u_{2,3} + 2c_{24}u_{3,2} + (c_{23} + c_{44})u_{3,3}] + w_{2,3}[c_{34}u_{2,3} + c_{44}u_{2,2}] + E^1_{11}[(c_{124} + \alpha(c_{24} + c_{224}))u_{2,2} + (c_{144} + \alpha(c_{44} + c_{244}))u_{2,3} + (c_{144} + \alpha c_{244})u_{3,2} + (c_{134} + \alpha c_{234})u_{3,3}] ,$$

$$D_2 = e_{22}u_{2,2} + e_{24}(u_{2,3} + u_{3,2}) - \epsilon_{22}\tilde{\hat{\varphi}}_{,2} - \epsilon_{23}\tilde{\hat{\varphi}}_{,3} . \tag{4.8}$$

A solution satisfying (4.7) and (4.4), with (4.8), may be taken in the usual form

$$u_\gamma = \sum_{m=1}^{3} C^{(m)} A_\gamma^{(m)} e^{i\beta_m \xi X_2} e^{i\xi(X_3 - Vt)} , \quad \gamma \neq 1$$

$$\tilde{\hat{\varphi}} = \sum_{m=1}^{3} C^{(m)} B^{(m)} e^{i\beta_m \xi X_2} e^{i\xi(X_3 - Vt)} , \tag{4.9}$$

where V, β_m, $A_\gamma^{(m)}$ and $C^{(m)}$ are to be determined numerically so that (4.7) and (4.4), with (4.8) are satisfied in the usual manner[15,18,19]. Calculations have been performed for Z-propagating surface waves on Y-cut lithium niobate[20,7], and the resulting lines are plotted in Fig.4. The notation on the lines in Fig.4 has the same meaning as in the case of quartz in Figs.2 and 3, and no a^* line is plotted.

5. Perturbation Procedure

If we define the change in phase velocity due to the applied flexural biasing state by

$$\Delta V = V_M - V \tag{5.1}$$

then it has been shown that for traction free and zero normal component of electric displacement boundary conditions the first perturbation in phase velocity is given by[9]

$$\Delta V = H_M / 2V_M \xi_M^2 , \tag{5.2}$$

where

$$H_M = - \int_0^\infty \int_0^{2\pi/\xi_1} [\hat{\tilde{K}}^n_{L\gamma} \hat{u}^M_{\gamma,L} + \hat{\tilde{\mathcal{D}}}^n_L \hat{\tilde{\varphi}}^M_{,L}]\, dX_2 dX_1 \tag{5.3}$$

$$\begin{aligned} \hat{\tilde{K}}_{L\gamma} &= \hat{c}_{L\gamma K\nu} \hat{u}^M_{\nu,K} + \hat{e}_{KL\gamma} \hat{\tilde{\varphi}}^M_{,K} , \\ \hat{\mathcal{D}}^n_L &= \hat{e}_{LK\alpha} \hat{u}^M_{\alpha,K} - \hat{\epsilon}_{LK} \hat{\tilde{\varphi}}^M_{,K} , \end{aligned} \tag{5.4}$$

with

$$\begin{aligned} \hat{u}^M_\gamma &= u^M_\gamma / N_M , \quad \hat{\tilde{\varphi}}^M = \tilde{\varphi}^M / N_M , \\ N_M^2 &= \rho^\circ \int_0^\infty \int_0^{2\pi/\xi_1} u^M_\gamma u^M_\gamma \, dX_2 dX_1 , \end{aligned} \tag{5.5}$$

where the coefficients in (5.4) are defined in (2.3). In the case of a purely elastic bias, which exists when electric and electroelastic coefficients are ignored, (2.11) and (2.12) hold in place of (2.3), and in place of (5.3) and (5.4) we have

$$H_M = -\int_0^\infty dX_2 \int_0^{2\pi/\xi_M} \hat{\tilde{K}}^n_L \hat{u}^M_{\gamma,L}\, dX_1 , \tag{5.6}$$

$$\hat{\tilde{K}}^n_{L\gamma} = \hat{c}_{L\gamma K\nu}\hat{u}^M_{\nu,K} , \tag{5.7}$$

where $\hat{c}_{L\gamma K\nu}$ is given in (2.11).

When (5.6) is applied to the case of rotated Y-cut quartz subject to cylindrical flexure in the propagation (X_1)-direction, we obtain the form

$$H_M = -\int_0^\infty dX_2 \int_0^{2\pi/\xi_M} [\hat{\tilde{K}}^n_{11}\hat{u}^M_{1,1} + \hat{\tilde{K}}^n_{12}\hat{u}^M_{2,1} + \hat{\tilde{K}}^n_{13}\hat{u}^M_{3,1} + \hat{\tilde{K}}^n_{21}\hat{u}^M_{1,2} + \hat{\tilde{K}}^n_{22}\hat{u}^M_{2,2} + \hat{\tilde{K}}^n_{23}\hat{u}^M_{3,2}]\, dX_1 . \tag{5.8}$$

The substitution of (5.7), with (2.11), along with the appropriate relations in Sec.3 into (5.8) yields

$$\begin{aligned} H_M = -\int_0^\infty dX_2 \int_0^{2\pi/\xi_M} dX_1 [& (c^*_{11} + 2c_{11} + c_{111} + \alpha c_{112})E^1_{11}\hat{u}_{1,1}\hat{u}_{1,1} + (c^*_{11} + c_{166} \\ & + \alpha(2c_{66} + c_{266}))E^1_{11}\hat{u}_{2,1}\hat{u}_{2,1} + (c^*_{11} + c_{155} + \alpha c_{255})E^1_{11}\hat{u}_{3,1}\hat{u}_{3,1} + (2c_{66} \\ & + c_{166} + \alpha c_{266})E^1_{11}\hat{u}_{1,2}\hat{u}_{1,2} + (c_{122} + \alpha(2c_{22} + c_{222}))E^1_{11}\hat{u}_{2,2}\hat{u}_{2,2} + (c_{144} \\ & + \alpha c_{244})E^1_{11}\hat{u}_{3,2}\hat{u}_{3,2} + 2(c_{12} + c_{112} + \alpha(c_{12} + c_{122}))E^1_{11}\hat{u}_{1,1}\hat{u}_{2,2} + 2(c_{14} \\ & + c_{114} + \alpha c_{124})E^1_{11}\hat{u}_{1,1}\hat{u}_{3,2} + 2(c_{66} + c_{166} + \alpha(c_{66} + c_{266}))E^1_{11}\hat{u}_{1,2}\hat{u}_{2,1} \\ & + 2(c_{56} + c_{156} + \alpha c_{256})E^1_{11}\hat{u}_{1,2}\hat{u}_{3,1} + 2(c_{156} + \alpha(c_{56} + c_{256}))E^1_{11}\hat{u}_{3,1}\hat{u}_{2,1} \\ & + 2(c_{124} + \alpha(c_{24} + c_{224}))E^1_{11}\hat{u}_{2,2}\hat{u}_{3,2} + 2c_{66}w_{1,2}\hat{u}_{1,1}\hat{u}_{2,1} + 2c_{22}w_{1,2}\hat{u}_{1,2}\hat{u}_{2,2} \\ & + 2c_{56}w_{1,2}\hat{u}_{1,1}\hat{u}_{3,1} + 2c_{24}w_{1,2}\hat{u}_{1,2}\hat{u}_{3,2} + 2(c_{12} + c_{66})w_{1,2}\hat{u}_{1,1}\hat{u}_{1,2} \\ & + 2c_{11}w_{2,1}\hat{u}_{1,1}\hat{u}_{2,1} + 2c_{66}w_{2,1}\hat{u}_{1,2}\hat{u}_{2,2} + 2c_{14}w_{2,1}\hat{u}_{3,2}\hat{u}_{2,1} \\ & + 2c_{56}w_{2,1}\hat{u}_{2,2}\hat{u}_{3,1} + 2(c_{12} + c_{66})w_{2,1}\hat{u}_{2,1}\hat{u}_{2,2}] , \end{aligned} \tag{5.9}$$

where from (3.10) and (3.11), with Fig.1

$$E_{11}^{(1)} = \left(1 - \frac{X_2}{h}\right) E^S, \quad w_{2,1} = \frac{X_1}{h} E^S, \quad w_{1,2} = -\frac{X_1}{h} E^S, \tag{5.10}$$

and

$$E^S = -\,3M/2h^2 c_{11}^{*}. \tag{5.11}$$

When the integrals in (5.9) are performed, the resulting expression is extremely cumbersome and not terribly revealing. Consequently, we do not present the complete equation from which the calculations are made. However, in order to give some idea of the type of terms that occur, we present two typical distinct types of integrals. These are

$$\begin{aligned}\int_0^\infty dX_2 \int_{-\pi/\xi_M}^{\pi/\xi_M} dX_1\, E_{11}^1 \hat{u}_{1,1}\hat{u}_{1,1} &= \int_0^\infty dX_2 \int_{-\pi/\xi_M}^{\pi/\xi_M} dX_1 \left(1 - \frac{X_2}{h}\right) E^S \hat{u}_{1,1}\hat{u}_{1,1} \\ = \mathrm{Re}\left[i\pi \sum_{m=1}^{4}\sum_{n=1}^{4} \frac{C^{(m)}A_1^{(m)}C^{(n)*}A_1^{(n)*}}{N_M^2(\beta_m - \beta_n^*)}\right] &+ \frac{\pi}{\xi_M h} \sum_{m=1}^{4}\sum_{n=1}^{4} \frac{C^{(m)}A_1^{(m)}C^{(n)*}A_1^{(n)*}}{N_M^2(\beta_m - \beta_n^*)^2}\Bigg] E^S,\end{aligned} \tag{5.12}$$

$$\begin{aligned}\int_0^\infty dX_2 \int_{-\pi/\xi_M}^{\pi/\xi_M} w_{2,1}\hat{u}_{1,2}\hat{u}_{2,2}\, dX_1 &= \int_0^\infty dX_2 \int_{-\pi/\xi_M}^{\pi/\xi_M} dX_1 \frac{X_1}{h} E^S \hat{u}_{1,2}\hat{u}_{2,2} \\ = \mathrm{Re}\,\frac{\pi}{2\xi_M}\Bigg[-\sum_{m=1}^{4}\sum_{n=1}^{4} & \frac{C^{(m)}A_1^{(m)}C^{(n)}A_2^{(n)}\beta_m\beta_n}{N_M^2(\beta_m + \beta_n)} \\ &+ i2\pi \sum_{m=1}^{4}\sum_{n=1}^{4} \frac{C^{(m)}A_1^{(m)}C^{(n)*}A_2^{(n)*}\beta_m\beta_n^*}{N_M^2(\beta_m - \beta_n^*)}\Bigg] \frac{E^S}{h},\end{aligned} \tag{5.13}$$

where

$$N_M^2 = \frac{2\pi\rho_o i}{\xi_M^2} \sum_{m=1}^{4}\sum_{n=1}^{4} \frac{C^{(m)}A_\gamma^{(m)}C^{(n)*}A_\gamma^{(n)*}}{(\beta_m - \beta_n^*)}, \tag{5.14}$$

the * denotes complex conjugate and 2h is the thickness of the plate, which is many times larger than the wavelength of the surface wave so that

$$(u_\gamma, \tilde{\varphi}) \rightarrow 0 \text{ at } X_2 = 2h . \tag{5.15}$$

Note that the biasing term in the typical integral in (5.13) is the type of term that was ignored in the direct calculations performed in Sec.4. When all terms of this type, i.e., those not giving rise to a strain, are ignored along with the X_2-dependence of the strain E^1_{11} in the perturbation calculations for both ST-cut and Y-cut quartz, the results are indistinguishable from the lines labeled a in Figs.2 and 3, respectively. However, when the spatial variation of all biasing terms in (5.9) is included in the perturbation calculations for ST-cut and Y-cut quartz, the resulting curves plot as the lines labeled a′ in Figs.2 and 3, respectively. It can be seen from the figures that although the difference between curves a and a′ is small, for relatively thick substrates, it increases appreciably with decreasing thickness-to-wavelength ratio. This means that since the change in such physical quantities as phase velocity with biasing deformation is very small, the perturbation procedure is significantly more accurate than the direct calculation for the determination of such quantities because it enables the influence of variable biasing terms to be included in the calculation.
When perturbation calculations are performed for ST-cut and Y-cut quartz subject to cylindrical flexure in the X_3-direction, i.e., normal to the propagation direction, the resulting curves plot as the curves labeled b′ in Figs.2 and 3, respectively. In this case no terms due to the variable biasing diaplacement gradients $w_{2,3}$ and $w_{3,2}$ survive in the perturbation calculation because they are odd in X_3 and the surface wave is independent of X_3.

When (5.6) is applied to the case of Y-cut lithium niobate subject to cylindrical flexure in the propagation (X_3)-direction, we obtain the form

$$H_M = -\int_0^\infty dX_2 \int_{-\pi/\xi_M}^{\pi/\xi_M} [K^n_{22}u_{2,2} + K^n_{23}u_{3,2} + K^n_{32}u_{2,3} + K^n_{33}u_{3,3}]\, dX_3 . \tag{5.16}$$

The substitution of (5.7), with (2.11), along with the appropriate relations in Sec.3 into (5.16) yields

$$H_M = - \int_0^\infty dX_2 \int_{-\pi/\xi_M}^{\pi/\xi_M} dX_3 [(c_{223} + \beta(2c_{22} + c_{222}))E^1_{33}\hat{u}_{2,2}\hat{u}_{2,2} + 2(c_{234} + \beta(2c_{24}$$
$$+ c_{224}))E^1_{33}\hat{u}_{2,2}\hat{u}_{2,3} + 2(c_{24} + c_{234} + \beta(c_{24} + c_{224}))E^1_{33}\hat{u}_{2,2}\hat{u}_{3,2} + 2(c_{23} + c_{233}$$
$$+ \beta(c_{23} + c_{223}))E^1_{33}\hat{u}_{2,2}\hat{u}_{3,3} + (c^*_{33} + c_{344} + \beta(2c_{44} + c_{244}))E^1_{33}\hat{u}_{2,3}\hat{u}_{2,3} + 2(c_{44}$$
$$+ c_{344} + \beta(c_{44} + c_{244}))E^1_{33}\hat{u}_{2,3}\hat{u}_{3,2} + 2(c_{334} + \beta c_{234})E^1_{33}\hat{u}_{2,3}\hat{u}_{3,3} + (2c_{44} + c_{344}$$
$$+ \beta c_{244})E^1_{33}\hat{u}_{3,2}\hat{u}_{3,2} + 2(c_{334} + \beta c_{234})E^1_{33}\hat{u}_{3,2}\hat{u}_{3,3} + (c^*_{33} + 2c_{33} + c_{333}$$
$$+ \beta c_{233})E^1_{33}\hat{u}_{3,3}\hat{u}_{3,3} - 2c_{24}w_{3,2}\hat{u}_{2,2}\hat{u}_{2,2} - 2(c_{23} + c_{44})w_{3,2}\hat{u}_{2,2}\hat{u}_{2,3} + 2(c_{22}$$
$$- c_{44})w_{3,2}\hat{u}_{2,2}\hat{u}_{3,2} + 2c_{24}w_{3,2}\hat{u}_{2,2}\hat{u}_{3,3} + 2c_{24}w_{3,2}\hat{u}_{2,3}\hat{u}_{3,2} + 2(c_{44}$$
$$- c_{33})w_{3,2}\hat{u}_{2,3}\hat{u}_{3,3} + 2c_{24}w_{3,2}\hat{u}_{3,2}\hat{u}_{3,2} + 2(c_{23} + c_{44})w_{3,2}\hat{u}_{3,2}\hat{u}_{3,3}] , \quad (5.17)$$

where from (3.14) and (3.15), with Fig.1, we have

$$E^1_{33} = \left(1 - \frac{X_2}{h}\right)\tilde{E}^S, \quad w_{2,3} = \frac{X_3}{h}\tilde{E}^S, \quad w_{3,2} = -\frac{X_3}{h}\tilde{E}^S, \quad (5.18)$$

and

$$\tilde{E}^S = - 3M/2h^2c^*_{33} . \quad (5.19)$$

As in the case of quartz and for the same reasons we do not present the full equations from which the calculations were made, but instead present the two typical distinct types of integrals

$$\int_0^\infty dX_2 \int_{-\pi/\xi_M}^{\pi/\xi_M} dX_3\, E^1_{33}\hat{u}_{2,2}\hat{u}_{2,2} = \int_0^\infty dX_2 \int_{-\pi/\xi_M}^{\pi/\xi_M} dX_3 \left(1 - \frac{X_2}{h}\right) \tilde{E}^S \hat{u}_{2,2}\hat{u}_{2,2}$$

$$= \mathrm{Re}\left[i\pi \sum_{m=1}^{3} \sum_{n=1}^{3} \frac{C^{(m)}A_2^{(m)}C^{(n)*}A_2^{(n)*}\beta_m\beta_n^*}{N_M^2(\beta_m - \beta_n^*)} \right.$$

$$\left. + \frac{\pi}{\xi_M h} \sum_{m=1}^{3} \sum_{n=1}^{3} \frac{C^{(m)}A_2^{(m)}C^{(n)*}A_2^{(n)*}\beta_m\beta_n^*}{N_M^2(\beta_m - \beta_n^*)^2} \right] \tilde{E}^S \,, \tag{5.20}$$

$$\int_0^\infty dX_2 \int_{-\pi/\xi_M}^{\pi/\xi_M} dX_3\, w_{2,3}\hat{u}_{2,2}\hat{u}_{3,2} = \int_0^\infty dX_2 \int_{-\pi/\xi_M}^{\pi/\xi_M} dX_3 \left(\frac{X_3}{h}\, \hat{u}_{2,2}\hat{u}_{3,2}\right) \tilde{E}^S$$

$$= \mathrm{Re}\, \frac{\pi}{2\xi_M} \left[- \sum_{m=1}^{3} \sum_{n=1}^{3} \frac{C^{(m)}A_2^{(m)}C^{(n)}A_3^{(n)}\beta_m\beta_n}{N_M^2(\beta_m + \beta_n)} \right.$$

$$\left. + i2\pi \sum_{m=1}^{3} \sum_{n=1}^{3} \frac{C^{(m)}A_2^{(m)}C^{(n)*}A_3^{(n)*}\beta_m\beta_n^*}{N_M^2(\beta_m - \beta_n^*)} \right] \frac{\tilde{E}^S}{h} \,, \tag{5.21}$$

where

$$N_M^2 = \frac{2\pi\rho_o i}{\xi_M^2} \sum_{m=1}^{3} \sum_{n=1}^{3} \frac{C^{(m)}A_\gamma^{(m)}C^{(n)*}A_\gamma^{(n)*}}{(\beta_m - \beta_n^*)} \,. \tag{5.22}$$

The results of perturbation calculations for Y-cut, Z-propagating surface waves yield curves a′ and b′ in Fig.4.

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FIGURE CAPTIONS

Figure 1	Schematic Diagram of the Surface Wave Structure Showing the Applied Static Loading
Figure 2	Relative Change in Surface Wave Velocity Per Unit Applied Maximum Extensional Biasing Strain vs. Thickness-Wavelength Ratio for ST-X Quartz. The curves a' and b' are for cylindrical flexure in and normal to the direction of propagation, respectively.
Figure 3	Relative Change in Surface Wave Velocity Per Unit Applied Maximum Extensional Biasing Strain vs. Thickness-Wavelength Ratio for Y-X Quartz. The notation is the Same as in Fig.2.
Figure 4	Relative Change in Surface Wave Velocity Per Unit Applied Maximum Extensional Biasing Strain vs. Thickness-Wavelength Ratio for Y-Z Lithium Niobate. The notation is the same as in Fig.2.

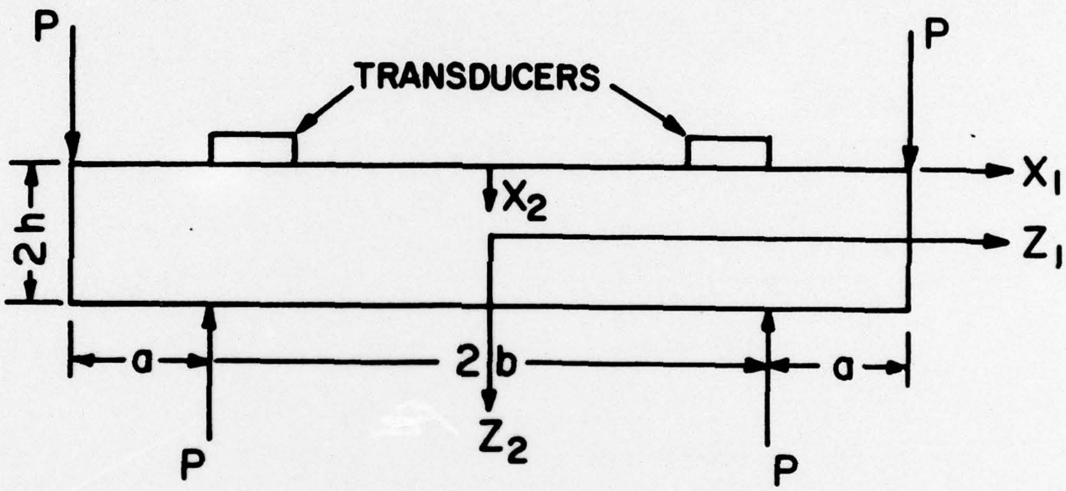

Figure 1

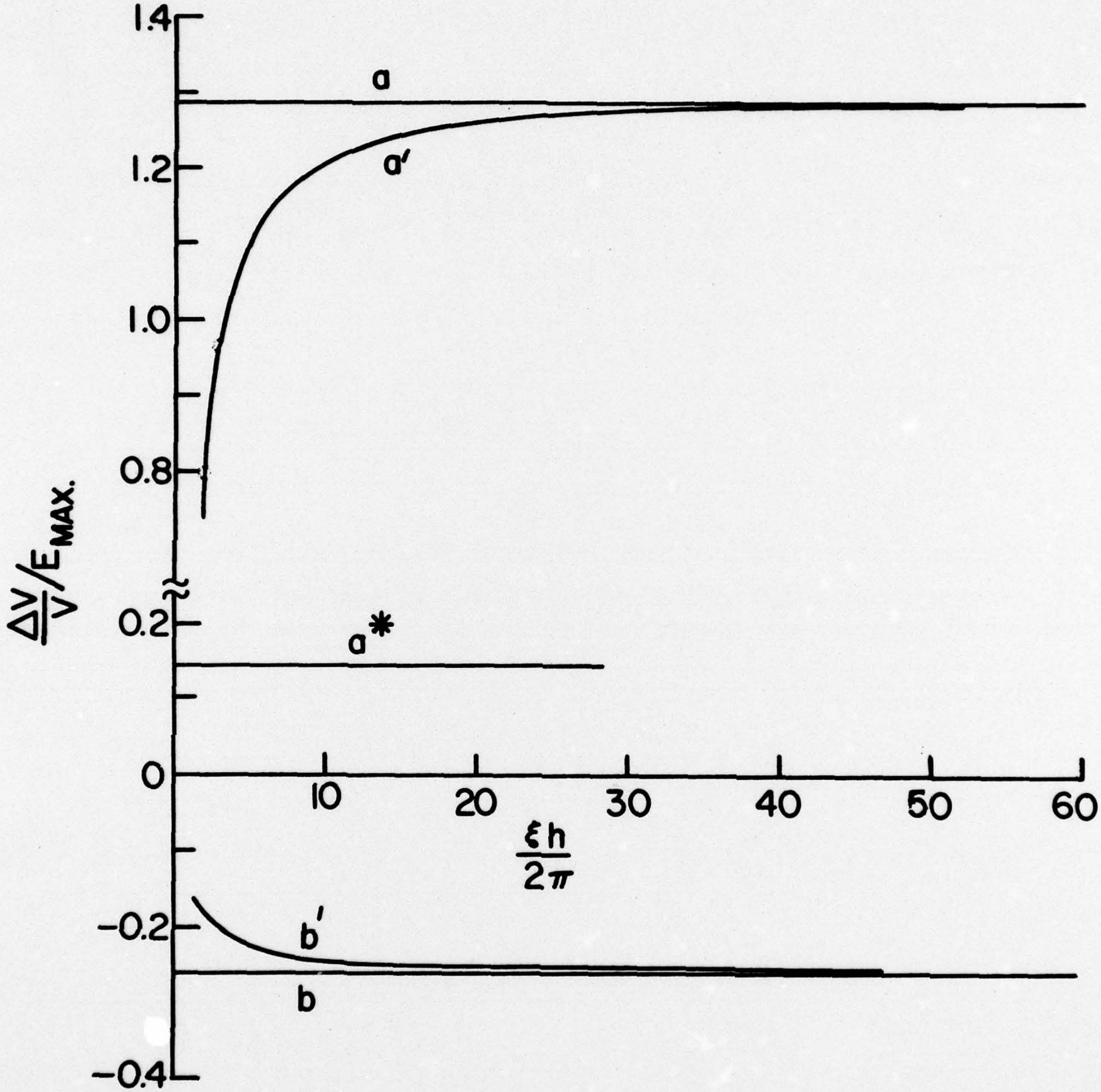
1.4
1.2
1.0
0.8
0.2
0
-0.2
-0.4
10
20
30
40
50
60
a
a′
a*
b′
b
$\frac{\Delta V}{V}/E_{MAX.}$
$\frac{\xi h}{2\pi}$

Figure 2

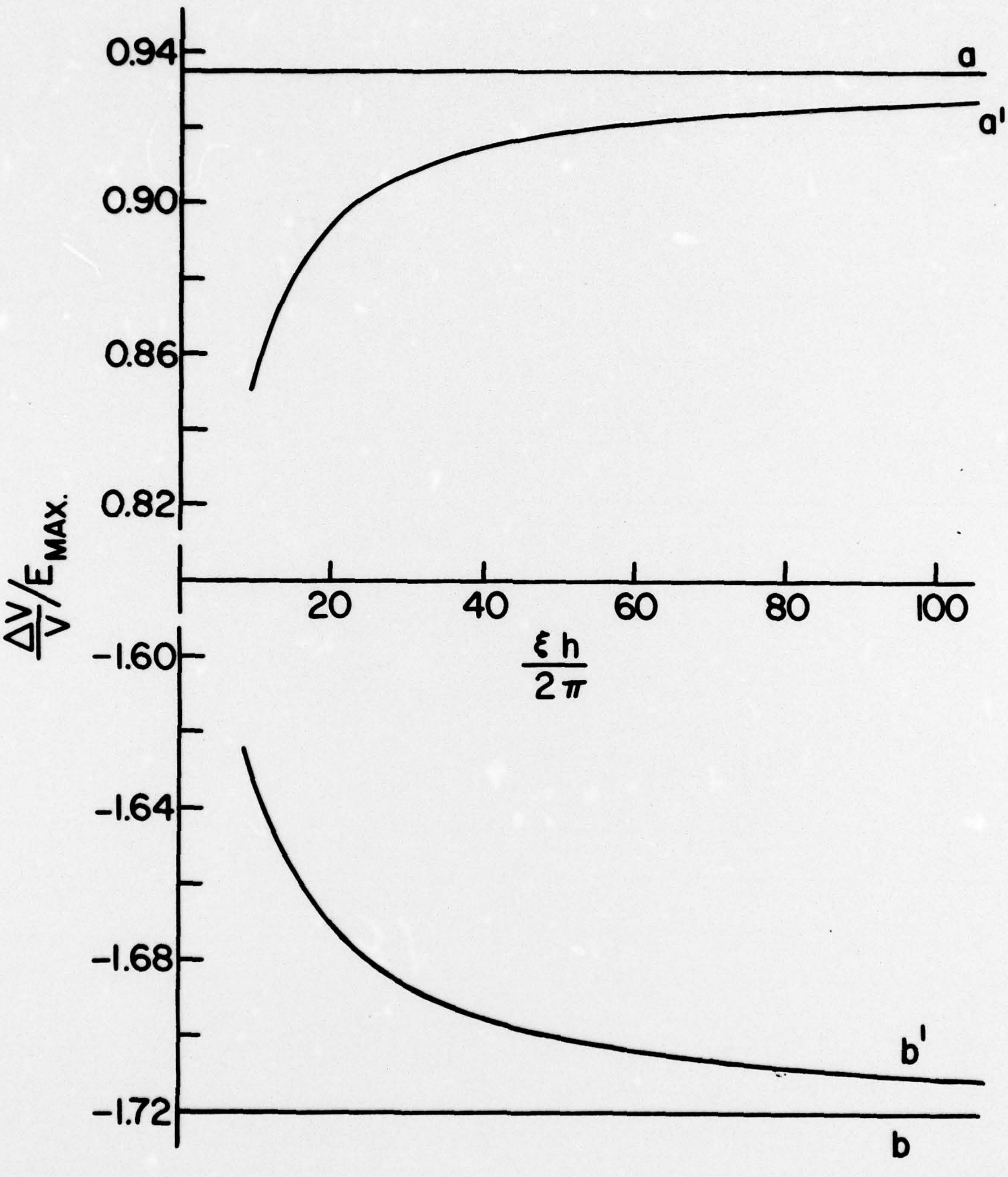
0.94
0.90
0.86
0.82
a
a'
20
40
60
80
100
$\frac{\xi h}{2\pi}$
$\frac{\Delta V}{V}/E_{MAX.}$
-1.60
-1.64
-1.68
-1.72
b'
b

Figure 3

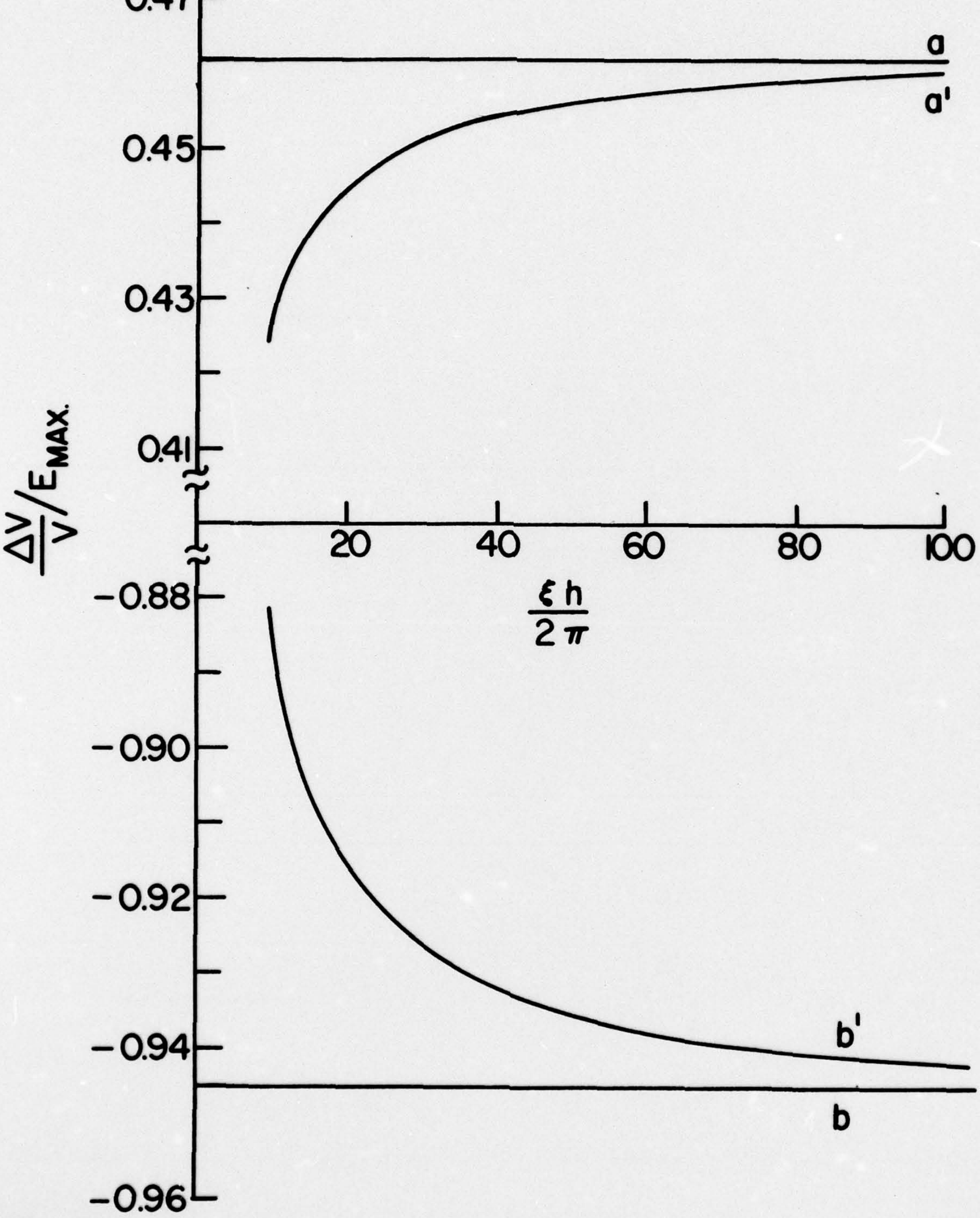
0.47
0.45
0.43
0.41
-0.88
-0.90
-0.92
-0.94
-0.96
$\frac{\Delta V}{V} / E_{MAX.}$
20
40
60
80
100
$\frac{\xi h}{2\pi}$
a
a'
b'
b

Figure 4